Sample Assignment #5: Lookin' for Love

While we would like to have all the necessary information at our disposal when we make a decision, in real life we do not always get that opportunity. For example, when you decide whether or not to marry someone, you have no idea who you will later meet in your life. Rather, the choice has to be made without knowing this information. An old tale encapsulates this problem rather nicely:

A sultan's advisor would like to marry. However, the sultan decrees that the advisor must meet 100 suitors, who will visit him in some random order. While each visits, they will present him with their dowry. Upon seeing this value, the advisor must choose to either take that woman, or not. If they do not, the next woman visits. To make things far more difficult, the sultan says that if the advisor doesn't choose the woman with the highest dowry, he will be executed. (This isn't an old tale to pass along to young children!) The question then becomes, what is the sultan's best strategy?

We will simplify the problem by assuming that the sultan will take on a strategy of the following form:

1) Let the first k women visit, rejecting each of them, noting down the highest dowry in that group.

2) As each subsequent woman comes, simply marry the next one who's dowry is higher than the previous maximum seen. (Obviously, if the woman with the true maximum dowry has already visited, the advisor is doomed and ends up picking the last woman who visits.)

The main question we will answer in this assignment is: what is the value of k the sultan should choose to optimize his chance of living (with the woman with the highest dowry!)

You will solve this problem in two steps:

1) Calculate the probability of success for a given value of k. (This probability will not be in closed form, rather, it will be a sum.

2) Write a computer program to cycle through all possible values of k and identify the k that leads to the maximum probability of success.

After this problem is solved, several other related problems will be posed.

Assume that the women are numbered 1, 2, 3, ..., 100, with the woman numbered 1 having the smallest dowry and the woman numbered 100 having the largest dowry. We can model the arrival of the women as a random permutation of 1, 2, 3, ..., 100.

Please give detailed justifications for each of your answers.

1) For a given number k, what is the total number of different *combinations* of women that could be the first k visitors out of 100?

2) Of all of those combinations determined in question 1, how many of those combinations have the highest value i ($100 \geq i \geq k$)? (ie. Think of the number of different combinations of k values such that i is the highest of the k values.)

3) Use your answer to questions 1 and 2 to determine the probability that out of the first k visitors, the visitor with number i is the highest visitor.

4) Given that the highest visitor of the first k visitor is the number i, what is the probability that the next visitor with a higher number is number 100? (Note: This question seems difficult, but remember that you ONLY have to consider those visitors with numbers greater than i, and each of them is equally likely to be in each relative position.)

5) Realizing that the highest visitor of the first k visitors could range anywhere from the woman numbered k to the woman numbered 99, for this technique to be successful, write a sum that represents the desired probability, the probability of success (finding the woman with the highest dowry) in terms of k.

6) Using this formula, write a computer program to calculate this probability for each value of k, starting from k=1 and ending at k=99. Identify the value of k that gives the greatest probability of success? For this value of k, what is the probability of success? (You may simply attach the whole chart of probabilities for each value of k, if you would like.

In real life, we don't need to marry the "best" woman in order to be happy. Instead of determining the probability of choosing the "best" woman, let's determine the probability of choosing one of the 10 best women using this strategy.

7) Adjust your sum from question #5 to solve this new problem. (Keep in mind that you might have to write out the sum of a couple different summations since terms may change.)

8) Adjust your computer program to calculate the optimal value of k for this new problem. What is the probability of success with this value of k?

9) Instead of being happy with one of the ten best women, perhaps you might have a different threshold. Try different values for this threshold. For each of these values that you try, find the optimal value of k and the corresponding probability of success.

10) What conclusions can you draw, (if any), both mathematically and about finding Mr. or Mrs. Right from your work? Is it possible that the idea here can extend to other areas?